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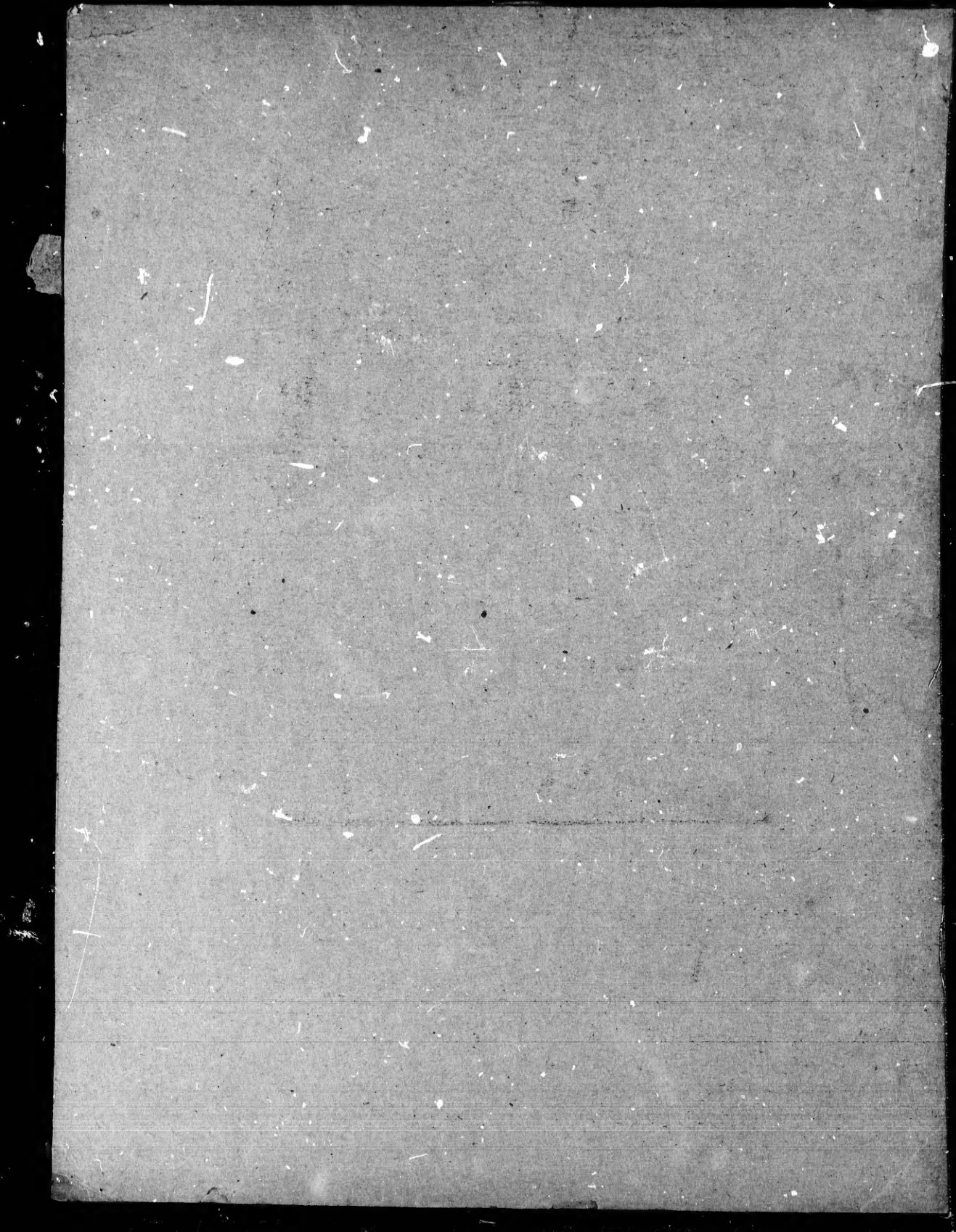
NO. 4.

ON THE SOLUTION OF A CERTAIN DIFFERENTIAL EQUATION  
WHICH PRESENTS ITSELF IN LAPLACE'S KINETIC THEORY  
OF TIDES.

By MR. GEORGE HERBERT LING, New York, N. Y.

[Submitted in partial fulfilment of the requirements for the degree of Doctor of Philosophy in the  
Faculty of Pure Science of Columbia University in the City of New York.]

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# ON THE SOLUTION OF A CERTAIN DIFFERENTIAL EQUATION WHICH PRESENTS ITSELF IN LAPLACE'S KINETIC THEORY OF TIDES.

By M<sup>r</sup>. GEORGE HERBERT LING, New York, N. Y.

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## I.

## INTRODUCTORY.

1. *Objects of the paper.* In his discussion of the kinetic theory of tides, Laplace found that the function expressing the height of the tide at a given point due to the attraction of the disturbing body satisfied a certain differential equation. Finding himself unable to obtain the general solution of the differential equation, he applied himself to the discussion of several particular cases which arise when certain assumptions are made regarding the physical constitution of the ocean. One of the cases he treated was that of the semi-diurnal tide when the depth of the ocean is supposed to be constant. In the course of his treatment of this case certain considerations enter which have given rise to much discussion. It is proposed to devote some attention to this case, and it is hoped to extend the treatment of this case so as to include some phases of it not previously treated. While it is generally conceded that the facts in regard to the disputed point referred to, have been made evident, yet the methods of placing those facts in evidence have been called into question by several writers on the subject, and do not appear to be the most satisfactory ones that are available.

## II.

## HISTORICAL SKETCH.

2. *Origin of the problem.* As just mentioned, the subject to be discussed was first treated by Laplace. His kinetic theory of tides is set forth in the *Mécanique Céleste*, and the part with which we are concerned is to be found in Livre IV of that work, his solution of the differential equation being given in Article 10. Considerable time had elapsed between his first discussion of the subject and the publication of his great work. The earliest presentation of his treatment of the subject was contained in a memoir\* presented to the Académie des Sciences, and contained in Tome IX of the *Œuvres de Laplace*. He has sought a solution of the equation in the form of a series of positive entire powers, and has made use of a certain infinite continued fraction in the evaluation of one of the coefficients of the series. The correctness of the value found by his method has been questioned. As the solution in the series form was made the basis of his calculations, it was of great importance that no mistake should be made in the determination of the coefficients, and more especially in the determination of those occurring early in the series.

\* Recherches sur plusieurs points du Systeme du Monde. Mémoires de l'Académie royale de Paris, année 1775-6.

3. *Previous contributions to the subject.* In his early memoir Laplace has gone somewhat more into detail, and the method by which he determined the value of the coefficient is clearly shown. In the later work he has omitted a great part of the explanation, and has contented himself with expressing the quantity in the form of the continued fraction to which reference has been made. The later presentation of the subject has been the more accessible of the two, and on it all the later writers appear to have based their remarks concerning Laplace's method, while the original presentation has been overlooked. Attention has been called to it by Prof. Lamb in his recent work on Hydrodynamics, and to him seems to be due the rediscovery, so to speak, of the memoir. Laplace's evaluation of the coefficient was objected to by Sir G. B. Airy,\* and later the same objection was made by Mr. William Ferrel.† A defence of Laplace was made by Lord Kelvin in the *Philosophical Magazine* for September, 1875. The October number of the same journal for 1875 contains a note written by Airy in which he reaffirms his objections to Laplace's result, and appears not to regard Kelvin's reasoning as convincing. The number of this journal for March, 1876, contains a reply by Ferrel to the arguments of Lord Kelvin. Prof. G. H. Darwin in the *Encyclopedia Britannica*‡ gives in more detail Lord Kelvin's argument. His treatment of the subject may also be found in Basset's *Hydrodynamics*, Vol. II, and Basset has briefly referred to the subject in a foot note.§ The latest contributions to the subject are believed to be the two papers by Ferrel which appear in Volumes 9 and 10 of Gould's *Astronomical Journal*. The latter of the two papers may also be found in the collection of papers|| on the "Mechanics of the Atmosphere" edited by Prof. Cleveland Abbe. Reference may also be made to Professor Lamb's *Hydrodynamics*, in which attention is called to Laplace's original memoir.

4. *Synopsis of previous contributions.* Before treating the problem analytically it will be useful to sketch the arguments of Laplace and those who afterwards treated the subject. Laplace, assuming that the solution of the equation could be expressed by means of a Taylor's series, substituted such a series with undetermined coefficients in the differential equation, and was able to determine all the coefficients of the series in terms of one of them, which remained arbitrary. He had previously argued that it was not necessary to obtain the general solution of the equation, since, as he affirmed, the arbitrary constants would be determined by the initial conditions of the water and would introduce effects dependent on this initial condition, which effects ought to be

\* Article, "Tides and Waves," *Encyclopedia Metropolitana*.

† "Tidal Researches," Appendix to United States Coast and Geodetic Survey Report for 1874.

‡ Article, "Tides," *Encyclopedia Britannica*.

§ Basset, Vol. II, p. 218.

|| No. 843, *Smithsonian Miscellaneous Contributions*.



disregarded, since in the case of the sea they would long ago have been overcome by friction. Considering, then, that any particular integral was sufficient, he proceeded to choose the most satisfactory value of the coefficient. His method of deciding the proper value of the coefficient will be given in Section III. It enabled him to satisfy himself that a comparatively few terms of his series would give the result with a very small error. From the form in which the result is set forth in the *Mécanique Céleste* it appears, however, that he made the assumption that the ratio of any coefficient in his series to the preceding one becomes ultimately smaller than any assignable quantity. Moreover Laplace's argument regarding the sufficiency of any particular solution did not occur immediately in connection with the treatment of this particular case, and it therefore appeared that he offered no justification for the assumption. Airy objected to the assumption on the ground that it was unnecessary and unduly specialized the solution. He added that, if the sea were bounded by a parallel of latitude instead of covering the whole earth, then the arbitrary constant could be determined from the corresponding boundary condition. Ferrel agreed with Airy, and regarding the constant as being entirely at his disposal, based his calculations on the series resulting from assigning to it the value zero. Kelvin in his reply to these arguments quoted Laplace's reasoning regarding the sufficiency of a particular solution, but pointed out that this reasoning was not correct. Proceeding, he contended that, as demanded by Airy, there was a certain physical condition to be satisfied, and that this condition was sufficient to justify Laplace's result. He also pointed out that the general solution of the equation should contain two arbitrary constants, and that a further boundary condition would be necessary for the determination of the second of these constants. He showed that, since the oscillations of the water which are taken account of in the differential equation have a perfectly definite period depending on the period of the disturbing body, the original state of motion could not be taken account of in the solution, for, except for special depths of water, the period of the latter oscillations would be different from that of the former. Airy and Ferrel, however, did not admit the force of the reasoning by means of which Kelvin justified Laplace's result, and Ferrel's later papers are devoted to an attempt to show that the determination of the value of the constant is unnecessary. While it would seem that the constant was correctly determined by Laplace, it appears to the author that the analytical proof of this fact indicated by Lord Kelvin, and given in greater detail by Prof. Darwin, is not complete. It seems, too, to be desirable to obtain the general solution of the differential equation, and to follow up the suggestions of Lord Kelvin regarding the application of this solution to the more general case and some special cases.

## III.

## LAPLACE'S SOLUTION.

5. *General outline.* Apart from physical considerations, the arguments made by Laplace from analysis do not appear to offer a very good reason for his evaluation of the coefficient. In order to show this, it will be necessary to give Laplace's discussion as it appeared in his original memoir. The form of the equation as treated by Darwin differs slightly from that in which Laplace used it, but no essential difference is introduced by the change, and the same difficulties arise in both cases. As Darwin's form of the equation is doubtless that in which the equation will henceforth be studied, it has been adopted here. The equation may then be written

$$x^2(1-x^2)\frac{d^2u}{dx^2} - x\frac{du}{dx} - (8-2x^2-\beta x^4)u + E\beta x^6 = 0. \quad (1)$$

It is to be noted that  $x$  is the sine of the polar distance of a particle of water,  $u$  the difference between the tide height in the dynamical theory and the tide height in the equilibrium theory. Assuming as the solution of (1) a Taylor's series containing only even powers and with undetermined coefficients, Laplace found that the coefficient of  $x^4$  remained undetermined. He next proceeded as follows: He assumed as an integral a sum of a finite number of powers, and found that by adding a certain term to the left member of (1) this equation could be modified so as to have as a solution the assumed function. By studying the effect of increasing the number of terms in this function, he came to the conclusion that a very small error would be made in assuming as a solution, such a function with a large number of terms.

6. *Application.* To apply this treatment to the equation (1), assume that

$$u = A_0 + (A_1 - E)x^2 + \sum_{k=2}^{r+1} A_k x^{2k}. \quad (2)$$

If this function satisfies equation (1), the following relations must be satisfied by the coefficients:

$$A_0 = 0, \quad (a)$$

$$A_1 - E = 0, \quad (b)$$

$$A_2 = A_2, \quad (c)$$

$$16A_3 - 10A_2 + \beta A_1 = 0, \quad (d)$$

$$2A_{k+1}[2(k-1)^2 + 6(k-1)] - 2A_k[2(k-1)^2 + 3(k-1)] + \beta A_{k-1} = 0, \quad (e)$$

$$(k = 3, 4, 5, \dots, r.)$$

$$-2A_{r+1}(2r^2 + 3r) + \beta A_r = 0, \quad (f)$$

$$+ \beta A_{r+1} = 0. \quad (g)$$





so that the expression (2) is a solution of equation (1) thus modified. The new equation can be written

$$x^2(1-x^2)\frac{d^2u}{dx^2} - x\frac{du}{dx} - u(8-2x^2-\beta x^4) + E\beta x^6 - \frac{E\mu^r}{\mu_1\mu_2\ldots\mu_r}x^{2r+n} = 0. \quad (3)$$

Laplace then argued that if the corrective term were very small, only a small error would be made if (1) were replaced by (3). He then proceeded to show that by taking  $r$  large enough the corrective term could be made to decrease indefinitely.

7. *Objections to method.* In order that the discussion just given may justify the choice of the value of  $A_2$ , it should put in evidence some property possessed by the series when Laplace's value is given to  $A_2$ , and not possessed by it under other circumstances. All that is attempted in the preceding process is to show that the error made in assuming as the integral a finite number of terms belonging to the infinite series can be made less than any assignable quantity. But this is true of any series which is an integral of the equation and whose region of convergence is large enough to suit the conditions of the problem, and it will appear that no matter what value be given to  $A_2$  the region of convergence of the resulting series is still large enough to suit the purpose. Moreover, it is not definitely proven that a vanishing correction to the equation necessarily indicates a corresponding vanishing of the error due to the assumption of a finite number of terms instead of an infinite number.

8. *Relation between correction to equation and error in series.* The relation between the correction to the equation and the error in the value of the dependent variable can be shown perhaps more clearly in the following manner, assuming certain properties of the series used which will be deduced later: The differential equation (1) can be regarded as a linear relation connecting the quantities  $u$ ,  $\frac{du}{dx}$ , and  $\frac{d^2u}{dx^2}$  in which the coefficients are rational entire functions of  $x$ . The function  $u$  is to be expressed by means of an infinite series. This series will have a certain circle of convergence. For all points *within* this circle the corresponding series for  $\frac{du}{dx}$  and  $\frac{d^2u}{dx^2}$  will also converge. Suppose the circumference of this circle of convergence lies entirely outside of the boundary of the region in which the independent variable is to vary. Then when  $u$ ,  $\frac{du}{dx}$ , and  $\frac{d^2u}{dx^2}$  are each replaced by the finite number of terms from the series expressing their values, certain errors will be made in the case of each. These errors will each become less than any assignable quantity if the number of terms be sufficiently increased. Let (1) then be written

$$a\frac{d^2u}{dx^2} + \beta_1\frac{du}{dx} + \gamma u = \delta,$$

where  $a$ ,  $\beta_1$ ,  $\gamma$ , and  $\delta$  are finite for all values to be considered. Let the true values of  $\frac{d^2u}{dx^2}$ ,  $\frac{du}{dx}$ , and  $u$  when found from the infinite series be  $A$ ,  $B$ , and  $C$ ; and let  $\varepsilon_1$ ,  $\varepsilon_2$ , and  $\varepsilon_3$  be the corresponding errors made in taking the finite number of terms for each of the three quantities. Then

$$aA + \beta_1 B + \gamma C = \delta, \quad (h)$$

and

$$a(A + \varepsilon_1) + \beta_1(B + \varepsilon_2) + \gamma(C + \varepsilon_3) \geq \delta. \quad (k)$$

Then the correction to be added to the left member of (k) to make it an equality is  $-(a\varepsilon_1 + \beta_1\varepsilon_2 + \gamma\varepsilon_3)$ . Under the circumstances assumed above, in reference to the region of convergence, this correction will become indefinitely small when the number of the terms is sufficiently increased. But other cases may occur. Consider the case of the infinite series in which  $A_2$  is given any other value than that assigned by Laplace. It will afterwards appear that this series converges for all values of  $x$  within the unit circle, and also for  $x = \pm 1$ . The series for  $\frac{du}{dx}$  and  $\frac{d^2u}{dx^2}$ , however, are convergent only for points within the unit circle. It is clear then that, for all values of  $x$  less than unity,  $\varepsilon_1$ ,  $\varepsilon_2$ , and  $\varepsilon_3$  can be made less than any assignable quantity, and that therefore the same is true for  $a\varepsilon_1 + \beta_1\varepsilon_2 + \gamma\varepsilon_3$ .

But  $x$  must be considered for all values up to and including unity, and it does not appear that, as  $x$  approaches unity, the correction must necessarily indefinitely diminish, since  $\varepsilon_3$  is the only one of the three quantities which indefinitely diminishes. Moreover the series is a satisfactory solution if only  $\varepsilon_3$  can be made indefinitely small for  $x = 1$ . On the other hand, it would also appear that the correction might be evanescent when  $\varepsilon_3$  is not so.

9. *Assumption equivalent to Laplace's assumption.* The same set of equations for the determination of the coefficients will be obtained if it be assumed that

$$\sum_{r=0}^{\infty} \frac{A_{r+1}}{A_r} = 0.$$

For then it would be proper to assume that the equations connecting the coefficients eventually took the same form as (f). For all other series in which the relation just written is not true the equation (f) is not satisfied when a finite number of terms is taken, but has to be corrected by the addition of a term. From this point of view then Laplace's process would seem to necessarily lead to a series convergent over the entire plane.

## IV.

## THE SOLUTION OF THE DIFFERENTIAL EQUATION.

10. *Character of the integral.* Laplace's solution having been considered, the general integral of the equation,

$$x^2(1-x^2) \frac{d^2u}{dx^2} - x \frac{du}{dx} - u(8-2x^2-\beta x^4) = -\beta E x^6, \quad (1)$$

may now be sought. In this equation it must be remembered that  $x = \sin \theta$  where  $\theta$  is a polar distance. It is necessary first to solve the auxiliary equation,

$$x^2(1-x^2) \frac{d^2u}{dx^2} - x \frac{du}{dx} - u(8-2x^2-\beta x^4) = 0. \quad (7)$$

The following general theorems will be useful:

THEOREM I. *In order that the equation*

$$\frac{d^n u}{dx^n} + p_1 \frac{d^{n-1}u}{dx^{n-1}} + p_2 \frac{d^{n-2}u}{dx^{n-2}} + p_3 \frac{d^{n-3}u}{dx^{n-3}} + \dots + p_{n-1} \frac{du}{dx} + p_n u = 0 \quad (8)$$

*shall have  $n$  independent integrals of the form,*

$$u = x^r (P_{n-1} \log^{n-1} x + P_{n-2} \log^{n-2} x + P_{n-3} \log^{n-3} x + \dots + P_1 \log x + P_0),$$

*where  $P_0, P_1, P_2, P_3, \dots, P_{n-1}$  are expressible in the neighborhood of  $x=0$ , in series of positive and negative powers of  $x$ , the number of negative powers in each being finite, it is necessary and sufficient that for each of the coefficients of the equation, such as  $p_0$ , the point  $x=0$  shall be an ordinary point, or a pole whose order of multiplicity does not exceed  $i$ .*

These integrals will constitute one or more groups of the form

$$\begin{aligned} u_1 &= x^r M_1, \\ u_2 &= x^r (M_2 \log x + N_2), \\ u_3 &= x^r (M_3 \log^2 x + N_3 \log x + R_3), \\ &\dots \dots \dots \\ u_k &= x^r (M_k \log^{k-1} x + N_k \log^{k-2} x + \dots), \end{aligned}$$

where  $M_1, M_2, M_3, \dots, M_k$  differ only by constant factors.

THEOREM II. *The integral of the equation (8) will be continuous and monogenic for all values of  $x$  for which the coefficients  $p_1, p_2, p_3, \dots, p_n$  are continuous and monogenic, and it can possess no critical points which are not*

also critical points of one or more of the coefficients. It may not have critical points at all the critical points of the coefficients.

The proofs of these theorems may be found in Jordan's *Cours d'Analyse*.\*

If, then, equation (7) be put in the form (8) and these theorems applied, it is clear that,

(1) Equation (7) has one integral which can be expressed in the form of a power series  $\psi$ , and a second integral which can be expressed in the form  $\psi_2 + \psi_1 \log x$ , where  $\psi_2$  and  $\psi_1$  are expressible as power series and  $\frac{\psi_1}{\psi} =$  a constant  $c$  (when  $c = 0$ , the second integral is also expressible as a power series);

(2) The general integral of (7) can have no critical points except at 0,  $\pm 1$ ,  $\pm \infty$ .

(3) If  $x$  be replaced by  $-x$ , the equation remains unaltered; whence it follows that the function has the same character at  $x = -a$  as at  $x = +a$ , and, in particular, if the general integral of (7) has a critical point of any sort at  $x = +1$  it has a critical point of exactly the same character at  $x = -1$ .

It follows, too, that if  $U$  is any particular integral of (1), the complete integral is

$$u = a_1\psi + a_2(\psi_2 + c\psi \log x) + U,$$

or

$$u = a_1\psi + a_2\psi_2 + U$$

when  $c = 0$ .

11. *Deduction of the complementary function.* It is correct, then, to assume as one integral,

$$u = \sum_r A_r x^{m+rs},$$

$s$  being a positive integer. Then

$$\frac{du}{dx} = \sum_r (m + rs) A_r x^{m-1+rs},$$

$$\frac{d^2u}{dx^2} = \sum_r (m + rs)(m - 1 + rs) A_r x^{m-2+rs},$$

The result of substituting in (7) is

$$\begin{aligned} & \sum_r (m + rs)(m - 1 + rs) A_r x^{m-2+rs} - \sum_r (m + rs)(m - 1 + rs) A_r x^{m+2+rs} \\ & - \sum_r (m + rs) A_r x^{m+rs} - \sum_r 8A_r x^{m+rs} \\ & + \sum_r 2A_r x^{m+2+rs} + \sum_r \beta A_r x^{m+1+rs} = 0, \end{aligned} \quad (9)$$

\* Vol. III, Arts. 146, 92, 118.

identically. The coefficient of  $x^m$ , the lowest power of  $x$ , must vanish; whence, since  $A_0 \geq 0$ ,

$$m^2 - 2m - 8 = 0, \text{ and } m = 4, -2.$$

The value of  $s$  must be 2, for if  $s \geq 2$  the coefficient of  $x^{m+2}$  does not vanish. The result of substituting  $s = 2$ ,  $m = 4$  in (9) is

$$\begin{aligned} & \sum_0^\infty (4 + 2r)(3 + 2r) A_r x^{2r+4} - \sum_0^\infty (4 + 2r)(3 + 2r) A_r x^{2r+6} \\ & - \sum_0^\infty (4 + 2r) A_r x^{2r+4} - \sum_0^\infty 8 A_r x^{2r+4} \\ & + \sum_0^\infty 2 A_r x^{2r+6} + \sum_0^\infty \beta A_r x^{2r+8} = 0. \end{aligned} \quad (10)$$

From (10) the equations for the determination of the coefficients are,

$$16A_1 - 10A_0 = 0, \quad (11)$$

$$2k(2k + 6) A_k - 2k(2k + 3) A_{k-1} + \beta A_{k-2} = 0, \quad (12)$$

$$(k = 2, 3, 4, 5, \dots).$$

Thus each coefficient is a multiple of  $A_0$  and one integral can be written

$$\begin{aligned} u_1 &= x^4 + \frac{A_1}{A_0} x^6 + \frac{A_2}{A_0} x^8 + \frac{A_3}{A_0} x^{10} + \dots \\ &= x^4 + C_1 x^6 + C_2 x^8 + C_3 x^{10} + \dots \end{aligned} \quad (13)$$

It is easy to verify that  $K_1 u_1$  is the series written down by Airy in his article in the Philosophical Magazine as the correction to Laplace's series,  $K_1$  being an arbitrary constant. On applying to the equation the theory set forth by Heffter,\* it appears that there is no series corresponding to the root  $m = -2$ . The second integral is then obtained in the form

$$u_2 = \psi_2 + \psi_1 \log x = \psi_2 + A' u_1 \log x.$$

From this

$$\begin{aligned} \frac{du_2}{dx} &= \frac{d\psi_2}{dx} + A' \log x \frac{du_1}{dx} + \frac{A'}{x} u_1, \\ \frac{d^2 u_2}{dx^2} &= \frac{d^2 \psi_2}{dx^2} + \frac{2A'}{x} \frac{du_1}{dx} - \frac{A'}{x^2} u_1 + A' \log x \frac{d^2 u_1}{dx^2}. \end{aligned}$$

\* Lineare Differential gleichungen, § 16.

These values being substituted in (7), there results

$$\begin{aligned} & x^2(1-x^2) \frac{d^2\psi_2}{dx^2} - x \frac{d\psi_2}{dx} - \psi_2(8-2x^2-\beta x^4) \\ & - A'u_1(2-x^2) + A' \cdot 2x(1-x^2) \frac{du_1}{dx} \\ & + A' \log x \left[ x^2(1-x^2) \frac{d^2u_1}{dx^2} - x \frac{du_1}{dx} - u_1(8-2x^2-\beta x^4) \right] = 0, \quad (15) \end{aligned}$$

identically. If the substitution  $\psi_2 = A'\psi_3$  be made and it be borne in mind that  $u_1$  is a solution of (7), (15) reduces to

$$\begin{aligned} & x^2(1-x^2) \frac{d^2\psi_3}{dx^2} - x \frac{d\psi_3}{dx} - \psi_3(8-2x^2-\beta x^4) \\ & - (2-x^2)u_1 + 2x(1-x^2) \frac{du_1}{dx} = 0. \quad (16) \end{aligned}$$

If now it be assumed that

$$\psi_3 = \sum_0^{\infty} B_r x^{m+r}, \quad (17)$$

and the substitution be made in (16), the equation for the determination of  $m$  is, as before,

$$m^2 - 2m - 8 = 0, \text{ whence } m = 4, -2.$$

The theory shows that  $m = -2$  is the root to be taken, and that, as before,  $s = 2$ . The substitution in (16) gives

$$\begin{aligned} & \sum_0^{\infty} (2r-2)(2r-3) B_r x^{2r-2} - \sum_0^{\infty} (2r-2)(2r-3) B_r x^{2r} \\ & - \sum_0^{\infty} (2r-2) B_r x^{2r-2} - \sum_0^{\infty} 8 B_r x^{2r-2} + \sum_0^{\infty} 2 B_r x^{2r} \\ & + \sum_0^{\infty} \beta B_r x^{2r+2} - \sum_0^{\infty} 2 C_r x^{2r+4} + \sum_0^{\infty} C_r x^{2r+6} \\ & + \sum_0^{\infty} 4(r+2) C_r x^{2r+4} - \sum_0^{\infty} 4(r+2) C_r x^{2r+6} = 0, \quad (18) \end{aligned}$$

in which the  $C$ 's are the coefficients in the series  $u_1$ .



From (19) are derived the relations

$$\begin{aligned}
 0 &= 8B_1 + 4B_0, \\
 0 &= 8B_2 - 2B_1 - \beta B_0, \\
 6 &= \quad \quad - \beta B_1, \\
 7 - 10C_1 &= 16B_1 - 10B_3 + \beta B_2, \\
 (4k - 5)C_{k-3} - (4k - 2)C_{k-2} &= (2k - 4)(2k + 2)B_{k+1} - (2k - 4)(2k - 1)B_k \\
 &\quad + \beta B_{k-1}. \\
 (k &= 4, 5, 6, \dots)
 \end{aligned} \tag{19}$$

These equations determine all the coefficients in terms of  $B_3$ , the coefficient of  $x^4$ , which remains arbitrary.

Since  $u_1$  is a solution of (7), it is clear that, if  $\varphi$  is a solution of (16), so also is  $\varphi + A'_3 u_1$ ,  $A'_3$  being any constant. Moreover  $u_1$  starts with the fourth power of  $x$ . Then if any value be assigned to  $B_3$  and the resulting value of  $\psi_3$  be denoted by  $\varphi$  we can write

$$\psi_3 = \varphi + A'_3 u_1.$$

$\varphi$  is a series of ascending entire powers starting from  $B_3 x^{-2}$  in which every coefficient is known.  $B_3$  may, if it is desired, be taken to be zero. The complete complementary function of (17) then is

$$\begin{aligned}
 u &= A'_1 (\psi_2 + \psi_1 \log x) + A'_2 u_1 \\
 &= A'_1 (\psi_2 + A'_1 u_1 \log x) + A'_2 u_1 \\
 &= A'_1 A'_1 (\psi_3 + u_1 \log x) + A'_3 u_1 \\
 &= A (\varphi + u_1 \log x) + B u_1.
 \end{aligned} \tag{20}$$

12. *The particular integral.* It remains to determine a particular integral of the complete equation (1). The character of  $u_1$  and of the absolute term of (1) makes clear the existence of a particular integral expressible in the form of a series of positive entire powers. Assume then

$$U = A_0 + (A_1 - E)x^2 + \sum_{r=2}^{\infty} A_r x^{2r}.$$

After substitution in (1) there result the relations

$$\begin{aligned}
 A_0 &= 0, \\
 A_1 - E &= 0, \\
 A_2 &= A_2, \\
 2A_{k+1} \{2(k-1)^2 + 6(k-1)\} - 2A_k \{2(k-1)^2 + 3(k-1)\} + \beta A_{k-1} &= 0. \\
 (k &= 2, 3, 4, \dots)
 \end{aligned}$$

All the coefficients are given in terms of  $E$  and  $A_2$ , the latter being arbitrary. Since only a particular integral is required any value may be given to  $A_2$ . In proceeding to a choice of a value it is interesting to follow the method by which Laplace's continued fraction is obtained. From the relations written just above, it can easily be deduced that

$$\begin{aligned} \frac{A_{k+1}}{A_k} &= 2(2k^2 + 3k) - 2(2k^2 + 6k) \frac{\beta}{A_{k+2}/A_{k+1}} \quad (k = 2, 3, 4, \dots) \\ \therefore \frac{A_2}{A_1} &= 2(2 \cdot 1^2 + 3 \cdot 1) - 2(2 \cdot 1^2 + 6 \cdot 1) \frac{\beta}{A_3/A_2} - 2(2 \cdot 2^2 + 6 \cdot 2) \frac{\beta}{A_4/A_3} - \dots \\ &\quad - 2(2(n-1)^2 + 6(n-1)) \frac{\beta}{A_{n+2}/A_{n+1}}, \\ &\quad 2[2(n-1)^2 + 3(n-1)] - 2[(2n^2 + 3n) - (2n^2 + 6n)] \frac{\beta}{A_{n+2}/A_{n+1}}, \end{aligned}$$

and  $A_1 = E$ .

The assumption,

$$\lim_{n \rightarrow \infty} \frac{A_{n+2}}{A_{n+1}} = 0, \quad (21)$$

gives the value of  $A_2$  in the form of an infinite continued fraction. It is permissible, if convenient, since any value of  $A_2$  may be taken. It was made by Laplace in the *Mécanique Céleste* apparently without justification; but, as has been seen, Laplace believed in the sufficiency of a particular solution and considered the resulting series as a satisfactory solution without the addition of the complementary function. The assumption (21) so affects the coefficients that the series converges for all finite values of  $x$ . It is not necessary that the particular integral should converge for points outside the unit circle. It is convenient from a mathematical point of view, however, to choose this series as the particular integral, for, if it were necessary to study the function for points outside the unit circle, it would be sufficient to obtain the complementary function in the form of a Laurent's series while the series just found would serve again as a particular integral. The assumption (21) is seen to be equivalent to that involved in Laplace's original process. From the reasoning of this section it is clear, too, that when

$$\lim_{n \rightarrow \infty} \frac{A_{n+1}}{A_n} = > 0,$$

then

$$\lim_{n \rightarrow \infty} \frac{A_{n+1}}{A_n} = 1.$$

From this point of view, too, it is clear that the series under consideration converges at least for every point within the unit circle, and that if it converges for a greater circle of convergence it converges for every finite value of  $x$ . For the sake of definiteness Laplace's value of  $A_2$  will be denoted by  $L$  and the series which furnishes the particular integral of (1) will be denoted by  $\varphi$ .

13. *Properties of the complete integral.* The complete integral of the equation (1) can then be expressed for points within the domain of the origin by

$$u = A (\varphi + u_1 \log x) + Bu_1 + \vartheta. \quad (22)$$

(1) Convergence:

The most general integral in the form of a positive power series can be written

$$u = B u_1 + \vartheta.$$

The relations among the coefficients of such a series show that, unless  $B$  is zero, the circle of convergence is of unit radius; and when  $B$  is zero, the circle of convergence has an indefinitely great radius.\* It follows, then, that  $u_1$  converges only for points within or on the circle of unit radius. Again,  $\varphi_3 = \varphi + A_2 u_1$  does not converge for points outside the unit circle. Then  $\varphi$  converges for points within the unit circle. It is conceivable that  $B_3$  may have been chosen so that  $\varphi$  shall converge all over the finite part of the plane.

(2) If  $A = B = 0$ , the function has no critical point except at infinity. If  $A = 0$ ,  $B \neq 0$ , the function has critical points at  $\pm 1$ ,  $\pm \infty$ . If  $B = 0$ ,  $A \neq 0$ , the function has a critical point at  $0$ ,  $\pm \infty$ , and (except for one particular choice of  $B_3$ ) at  $\pm 1$ . In addition to the singularity of  $\varphi$  at  $x = 0$  integrals of this class and integrals of the general class have a singularity at  $x = 0$  due to the singularity of  $\log x$  at that point, and are, in addition, many valued at any point owing to the properties of  $\log x$ . This indeterminateness will be removed if it is assumed that for positive real values of  $x$  the result shall be real. Again, when  $B = 0$ ,

$$\frac{du}{dx} = A \left[ \frac{d\varphi}{dx} + \log x \frac{du_1}{dx} + \frac{1}{x} u_1 \right] + \frac{d\vartheta}{dx}.$$

$$\text{But} \quad \frac{du}{d\theta} = \frac{du}{dx} \sqrt{1-x^2},$$

$$\begin{aligned} \oint_{\theta=\pi/2} \left[ \frac{du}{d\theta} \right] &= \oint_{x=1} \left[ \frac{du}{dx} \sqrt{1-x^2} \right] \\ &= A \oint_{x=1} \left[ \frac{1}{1-x^2} \frac{d\varphi}{dx} \right] + A \oint_{x=1} \left[ (\log x) \frac{1}{1-x^2} \frac{du_1}{dx} \right]. \end{aligned}$$

\* The detailed proof of this fact is quoted in Section VI.

It will afterwards be shown that for the complete integral (22)

$$\left[ \frac{du}{d\theta} \right]_{\theta=\pi/2} = \text{a finite quantity,}$$

and that

$$\left[ \frac{du_1}{d\theta} \right]_{\theta=\pi/2} = \text{a finite quantity.}$$

$\therefore$  for all integrals of the form

$$u = A(\varphi + u_1 \log x) + \Psi,$$

$$\left[ \frac{du}{d\theta} \right]_{\theta=\pi/2} = \text{a finite quantity or zero.}$$

V.

#### THE DETERMINATION OF THE CONSTANTS FOR LAPLACE'S CASE.

14. *The physical conditions.* It having been agreed that the constants shall be determined to suit the boundary conditions, the case discussed by Laplace, where the whole earth is covered with water, may now be treated. The expression for  $u$ , as given in (22), has an infinite value when  $x = 0$ , unless  $A = 0$ . Since there cannot be a tide of infinite depth at the pole it is necessary to make  $A = 0$ . The remaining expression is

$$u = Bu_1 + \Psi.$$

Airy and Ferrel contended that this was the exact expression for  $u$ , and that  $B$  could be given any value. Ferrel determined it by the condition

$$B + L = 0. \quad (23)$$

Lord Kelvin pointed out that owing to the symmetry of the disturbance in the two hemispheres the meridional displacement of water should vanish at the equator. The expression for the meridional displacement is the product of two terms of which one involves the latitude and the other does not. The factor involving the latitude is,

$$z = \frac{1}{4m \sin^2 \theta} \left[ \frac{du}{d\theta} + 2u \cot \theta \right]. \quad (24)$$

When  $\theta = \pi/2$ ,\*

$$z = \frac{1}{4m} \left[ \frac{du}{d\theta} \right]_{\theta=\pi/2}.$$

\*  $\theta$  = co-latitude or polar distance.

∴ at the equator it is necessary that

$$\frac{du}{d\theta} = 0. \quad (25)$$

But

$$\frac{du}{d\theta} = (1-x^2) \frac{du}{dx}.$$

∴ it is necessary that,

$$\sum_{r=1}^{\infty} \left\{ \sqrt{1-x^2} \frac{du}{dx} \right\} = 0. \quad (26)$$

15. *Proof that B = 0.* In order to prove that the condition (26) requires that *B* shall be zero, the function will be considered in the neighborhood of  $x = 1$ .

Assume

$$x = 1 + y; \quad (27)$$

then

$$\frac{du}{dx} = \frac{du}{dy}, \quad \frac{d^2u}{dx^2} = \frac{d^2u}{dy^2};$$

and equation (7) takes the form

$$(2y + 5y^2 + 4y^3 + y^4) \frac{d^2u}{dy^2} + (1+y) \frac{du}{dy} + u \{ (6-\beta) - 4(1+\beta)y - 2(1+3\beta)y^2 - 4\beta y^3 - \beta y^4 \} = 0. \quad (28)$$

Theorems I and II apply to (28) also. Then assume

$$u = \sum_{r=0}^{\infty} A_r y^{m+rs},$$

$s$  being a positive integer. The equation for the determination of  $m$  is

$$2m^2 - m = 0; \quad (29)$$

whence  $m = 0$  or  $+\frac{1}{2}$ . Also  $s = 1$ . The two integrals are expressible in series form. Moreover the relations connecting the  $A$ 's are, for both values of  $m$ , such that each coefficient is given as a multiple of the first one. The two independent integrals of (28) may then be written

$$y_1 = 1 + \sum_{r=1}^{\infty} \alpha_r y^r, \quad (30)$$

$$y_2 = y^{\frac{1}{2}} + \sum_{r=1}^{\infty} \beta_r y^{r+\frac{1}{2}},$$

the  $\alpha$ 's and  $\beta$ 's being known.

The complete integral of (28), then, is

$$u = c_1 y_1 + c_2 y_2. \quad (31)$$

If now the substitution (27) be made in (1), an equation results which differs from (28) only by the expression  $-E_1^2(1+y)^6$  in the right hand member. Of this equation, the complementary function is given by (31). To complete its integration, it is necessary to find a particular integral. The character of  $y_1$  and of the absolute term  $-E_1^2(1+y)^6$  make it clear that a particular integral can be obtained in the form of a series of positive entire powers. Let this integral be denoted by  $Y$ . It is not of importance that its coefficients be calculated. Then, in the neighborhood of  $x = 1$ , or of  $y = 0$ , the integral of (28) can be expressed in the form

$$u = c_1 y_1 + c_2 y_2 + Y.$$

Then

$$\frac{du}{dx} = \frac{du}{dy} = c_1 \frac{dy_1}{dy} + c_2 \frac{dy_2}{dy} + \frac{dY}{dy}.$$

Also

$$\sqrt{1-x^2} = \sqrt{-(y^2+2y)} = iy\sqrt{2+y},$$

where  $i = \sqrt{-1}$ .

$$\begin{aligned} \therefore \frac{du}{d\theta} = \frac{du}{dx} \sqrt{1-x^2} &= c_1 iy^{\frac{1}{2}}(2+y)^{\frac{1}{2}} \frac{dy_1}{dy} + c_2 iy^{\frac{1}{2}}(2+y)^{\frac{1}{2}} \frac{dy_2}{dy} \\ &+ iy^{\frac{1}{2}}(2+y)^{\frac{1}{2}} \frac{dY}{dy}. \end{aligned} \quad (32)$$

When  $y = 0$ , the first and third terms on the right vanish, and the middle term becomes

$$[+\frac{1}{2} y^{-\frac{1}{2}} c_2 iy^{\frac{1}{2}}(2+y)^{\frac{1}{2}}]_{y=0}, \text{ or } c_2 i/\sqrt{2}.$$

$$\therefore (du/d\theta)_{\theta=\pi/2} = c_2 i/\sqrt{2}. \quad (33)$$

In order, then, that at the equator  $du/d\theta = 0$ , the function must be such that  $c_2 = 0$ . But if  $c_2 = 0$ , the point  $x = 1$  is an ordinary point of the function, while if  $c_2 \neq 0$  the point  $x = 1$  is a branch point of the function. If the point  $x = 1$  is an ordinary point of the function, so also is  $x = -1$ .

In this case the function has no critical point in the finite part of the plane, and, if expressible in the neighborhood of the origin  $x = 0$  by means of a Taylor's series, that series will converge for all finite values of the variable  $x$ . If the point  $x = 1$  is a branch point of the function, and if the function is expressible in the neighborhood of the origin  $x = 0$  by means of a Taylor's series, that series will have a circle of convergence of unit radius. It follows,

then, when  $B = 0$  and  $u = \mathfrak{L}$ , that  $c_2 = 0$  and  $(du/d\theta)_{\theta=\pi/2} = 0$ , and when  $B > 0$ , that  $c_2 < 0$  and  $(du/d\theta)_{\theta=\pi/2} > 0$ .

Thus it is seen that the condition stated by Lord Kelvin requires that  $B = 0$  and  $u = \mathfrak{L}$ . Consequently the series as written by Laplace is the complete solution for the case of an earth completely covered to a constant depth by water.

NOTE.—Equation (33) gives an imaginary value for  $(du/d\theta)_{\theta=\pi/2}$  when the arbitrary constant  $c_2$  is taken real. It is evident, however, that, when real values of the function between  $x = 0$  and  $x = 1$  are desired,  $c_2$  must be taken purely imaginary; for it was assumed that

$$x = 1 + y; \quad (27)$$

so that when  $x$  is less than unity  $y$  is negative, and  $y^{\frac{1}{2}}$  is a pure imaginary. The  $c_2 y^{\frac{1}{2}}$  will be real when  $c_2$  is purely imaginary, and it follows that then  $y_2$  is also real.

## VI.

DARWIN'S PRESENTATION OF LORD KELVIN'S PROOF THAT  $B$  MUST BE ZERO WHEN  $(du/d\theta)_{\theta=\pi/2} = 0$ .

16. *Darwin's argument.* The function  $u = Bu_1 + \mathfrak{L}$  may be regarded as a single series of even positive integral powers commencing with the fourth and having the coefficient of  $x^4$  arbitrary. It has already been seen what relations connect the coefficients and define them in terms of the coefficient of  $x^4$  ( $A_2$  say). It is known, too, that when

$$\sum_{n=2}^{\infty} \frac{A_{n+1}}{A_n} = 0$$

then  $A_2 = L$ .

Suppose now that

$$\sum_{n=2}^{\infty} \frac{A_{n+1}}{A_n} < 0,$$

but  $= \alpha^{-1}$ , a finite quantity. Then

$$\begin{aligned} \frac{A_{n+2}}{A_{n+1}} &= \frac{2n+3}{2n+6} - \frac{\beta^2}{2n(2n+6)} \frac{A_n}{A_{n+1}} \\ &= \frac{2n+3}{2n+6} - \frac{\beta^2}{2n(2n+6)} (\alpha - h), \end{aligned} \quad (34)$$

where  $h$  tends to zero when  $n$  becomes indefinitely great. Darwin's argument is along the following lines: When

$$\sum_{n=c}^{\infty} \frac{A_{n+1}}{A_n} > 0$$

then, for large values of  $n$ ,

$$\frac{A_{n+2}x^{2n+4}}{A_{n+1}x^{2n+2}} = \frac{2n+3}{2n+6} x^2 = \left[1 - \frac{3}{2(n+3)}\right] x^2 = \left[1 - \frac{3}{2n}\right] x^2$$

nearly.

But if  $(1-x^2)^{\frac{1}{2}}$  be expanded by the binomial theorem the ratio of the  $(n+1)$ th term to the  $n$ th term is  $\left[1 - \frac{3}{2n}\right] x^2$ . Consequently, in this case, it is possible to write

$$u = A_1 + B_1(1-x^2)^{\frac{1}{2}},$$

where  $A_1$  and  $B_1$  are finite for all values of  $x$ . A similar argument being made in the case of the series for  $du/dx$  it follows that

$$du/dx = C + D(1-x^2)^{-\frac{1}{2}},$$

where  $C$  and  $D$  are finite (and not zero) for all values of  $x$ .

But

$$du/d\theta = du/dx (1-x^2)^{\frac{1}{2}} = C(1-x^2)^{\frac{1}{2}} + D;$$

$$\therefore (du/d\theta)_{\theta=\pi/2} = [C(1-x^2)^{\frac{1}{2}} + D]_{x=1} = D < 0.$$

17. *Discussion of Darwin's proof.* These results agree with each other, and with what has been proven in another way; but this proof of the fact that  $B_1$  and  $D$  are not zero nor infinite does not appear to be entirely satisfactory, and it is essential that this property of  $B_1$  and  $D$  be made evident. The ratio  $A_{n+2}/A_{n+1}$  becomes  $1 - \frac{3}{2}n^{-1}$  when the square and higher powers of  $n^{-1}$  be neglected. If after a certain value of  $n$  quantities of the order  $n^{-1}$  be neglected, the ratio  $A_{n+2}/A_n$  becomes unity. Then, following a line of argument similar to that given, it would appear that

$$u = A_2 + B_2(1-x^2)^{-1} \quad (a)$$

and, by a similar course of reasoning, that

$$du/dx = C_2 + D_2(1-x^2)^{-1}.$$

These results do not agree and are incorrect, but they show in what respect



the previous reasoning is weak. For, suppose that the binomial expansion of  $(1 - x^2)^{\frac{1}{2}}$  be written

$$(1 - x^2)^{\frac{1}{2}} = \sum_{r=0}^{\infty} \gamma_r x^{2r}.$$

Then, if the infinite series can be written in the form

$$u = A_1 + B_1 (1 - x^2)^{\frac{1}{2}},$$

where  $A_1$  and  $B_1$  are finite for all values of  $x$ , it follows that

$$B_1 = \sum_{r=\infty}^{\infty} \frac{A_r}{\gamma_r}.$$

Now for every finite value of  $n$  ( $j$  being positive),

$$\frac{A_{r+1}}{A_r} < \frac{\gamma_{r+1}}{\gamma_r}, \quad (b)$$

Darwin has shown that

$$\sum_{r=\infty}^{\infty} \frac{A_{r+1}}{A_r} = \sum_{r=\infty}^{\infty} \frac{\gamma_{r+1}}{\gamma_r};$$

but it is necessary also to show that

$$\sum_{r=\infty}^{\infty} A_1 \cdot \frac{A_2}{A_1} \cdot \frac{A_3}{A_2} \cdots \frac{A_r}{A_{r-1}} \cdot \frac{A_{r+1}}{A_r} \cdot \sum_{r=\infty}^{\infty} \frac{\gamma_1}{\gamma_1} \cdot \frac{\gamma_2}{\gamma_1} \cdot \frac{\gamma_3}{\gamma_2} \cdots \frac{\gamma_r}{\gamma_{r-1}} \cdot \frac{\gamma_{r+1}}{\gamma_r} \quad (c)$$

is finite.

Both numerator and denominator are zero so that the value of the quotient requires investigation.

The discussion of (a) and (c) shows that in (a)  $A_2$  and  $B_2$  do not have finite non-vanishing values for all values of  $x$ .

## VII.

18. *Cases to be treated.* It remains, then, to examine the other cases included in the solution obtained. Airy pointed out that in the solution of the form

$$u = Bu_1(x) + v(x),$$

$B$  could be determined so that the solution would be suitable for the case of a sea forming a spherical cap and extending from the pole to an arbitrary parallel of latitude. Lord Kelvin pointed out that, if the general solution were

at hand, the two constants could be determined so as to obtain a solution suitable for a zonal sea lying between two parallels of latitude. The most interesting cases to be dealt with appear then to be the following:—

1. *Case of a sea covering the whole earth. This is the case already treated.*
2. *Case of a sea extending from the pole to a given parallel of latitude.*
3. *Case of a zonal sea bounded by two parallels of latitude on opposite sides of the equator and equally distant from it.*
4. *Case of a zonal sea bounded by any two parallels of latitude lying in one hemisphere.*
5. *Case of a canal lying along a parallel of latitude.*

#### CASE 2.

19. *Polar sea.* If the sea extends only to a given parallel of latitude from the pole, it is necessary that the meridional component of the motion should vanish for the corresponding value of  $x$ , the sine of the polar distance of the boundary.

Then, as in Case 1,  $A = 0$ , and the condition just named gives for the boundary value of  $x$

$$\frac{du}{d\theta} + 2u \cot \theta = 0.$$

Let, then,  $\theta_1$  be the colatitude of the southern boundary, and  $\sin \theta_1 = a_1$ .

Since

$$\frac{du}{d\theta} = \frac{du}{dx} \cos \theta, \text{ and } \cos \theta_1 = 0,$$

$$\therefore \frac{du}{dx} + \frac{2u}{\sin \theta} = 0, \text{ for } \theta = \theta_1;$$

$$\therefore Bu_1'(a_1) + \mathfrak{V}'(a_1) + \frac{2}{a_1} \{ Bu_1(a_1) + \mathfrak{V}(a_1) \} = 0;$$

$$\therefore B = - \frac{a_1 \mathfrak{V}'(a_1) + 2\mathfrak{V}(a_1)}{a_1 u_1'(a_1) + 2u_1(a_1)}. \quad (35)$$

This gives the expression for  $u$  at any point in the form

$$u = \mathfrak{V}(x) - \frac{a_1 \mathfrak{V}'(a_1) + 2\mathfrak{V}(a_1)}{a_1 u_1'(a_1) + 2u_1(a_1)} u_1(x). \quad (36)$$

In particular, for the southern boundary,

$$u(a_1) = a_1 \cdot \frac{\mathfrak{V}(a_1) u_1'(a_1) - \mathfrak{V}'(a_1) u_1(a_1)}{a_1 u_1'(a_1) + 2u_1(a_1)}. \quad (37)$$

(36) and (37) give the amounts to be added to the tide deduced from the equilibrium theory.

The total tide at any point then is

$$U = u(x) + E x^2, \quad (38)$$

and at the southern boundary

$$U = u(a_1) + E a_1^2. \quad (39)$$

As has been stated, Ferrel's calculations were made for the series in which  $B$  was given by the relation

$$B + L = 0.$$

Then, from (35) it appears that the tides calculated by Ferrel would be those existing on a circumpolar sea bounded by a parallel of latitude  $\frac{1}{2}\pi - \theta$ , where  $\alpha = \sin \theta$  and satisfies the equation

$$L = \frac{\alpha \mathfrak{U}'(\alpha) + 2\mathfrak{U}(\alpha)}{\alpha u_1'(\alpha) + 2u_1(\alpha)}. \quad (39a)$$

#### CASE 3.

20. *Sea extending equally on both sides of equator.* Suppose the sea to extend equally on both sides of the equator, the boundaries being parallels of latitude.

The condition that there shall be no motion of water along the meridian at any point of the northern boundary gives one relation connecting  $A$  and  $B$ ; but it is clear that the corresponding condition for the southern boundary gives exactly the same relation; so that one of the constants appears to be arbitrary. The considerations which applied to Case 1 apply to this case. The symmetry of the motion requires that there be no meridional motion of the water at the equator. In this case, also, it is necessary that

$$(du/d\theta)_{\theta=\pi/2} = 0.$$

This gives a second condition by means of which the remaining arbitrary constant may be determined.

From

$$u = A \{ \varphi + u_1 \log x \} + B u_1 + \mathfrak{U} \quad (22)$$

it follows that

$$\frac{du}{d\theta} = A(1-x^2)^{\frac{1}{2}} \{ \varphi' + u_1' \log x + \frac{1}{x} u_1 \} + B(1-x^2)^{\frac{1}{2}} u_1' + (1-x^2)^{\frac{1}{2}} \mathfrak{U}'.$$

Now

$$\oint_{x=1} [(1-x^2)^{\frac{1}{2}} \chi'] = 0;$$

$$\oint_{x=1} \left[ \frac{(1-x^2)^{\frac{1}{2}}}{x} u_1 \right] = 0,$$

since  $u_1$  converges for  $x = 1$ ;<sup>\*</sup>

$$\oint_{x=1} [(1-x^2)^{\frac{1}{2}} u_1' \log x] = 0,$$

since  $\log 1 = 0$  and  $\oint_{x=1} [(1-x^2)^{\frac{1}{2}} u_1']$  is finite.

It is clear, then, that

$$A \oint_{x=1} [(1-x^2)^{\frac{1}{2}} \varphi'] + B \oint_{x=1} [(1-x^2)^{\frac{1}{2}} u_1] = 0.$$

It has been seen that

$$\oint_{x=1} [(1-x^2)^{\frac{1}{2}} u_1'] \text{ is a finite quantity, say } b;$$

$$\oint_{x=1} [(1-x^2)^{\frac{1}{2}} \varphi'] \text{ is a finite quantity, say } a.$$

(In one particular case it is possible that  $a$  might be zero.) Then it follows that

$$Aa + Bb = 0. \quad (40)$$

Returning now to the condition first stated, let  $\theta_1$  be the colatitude of the boundary, and let  $\sin \theta_1 = a_1$ . It is necessary that

$$\left[ \frac{du}{dx} + \frac{2u}{x} \right]_{x=a_1} = 0, \quad (41)$$

since  $\cos \theta_1 \geq 0$ . The resulting relation between  $A$  and  $B$  takes the form

$$A \left[ \varphi'(a_1) + \frac{2}{a_1} \varphi(a_1) + u_1'(a_1) \log a_1 + \frac{1}{a_1} u_1(a_1) + \frac{2}{a_1} u_1(a_1) \log a_1 \right] \\ + B \left[ u_1'(a_1) + \frac{2}{a_1} u_1(a_1) \right] + \chi'(a_1) + \frac{2}{a_1} \chi(a_1) = 0. \quad (42)$$

<sup>\*</sup> See equation (b) Section VI. The expansion of  $(1-x^2)^{\frac{1}{2}}$  converges for  $x = 1$ .

An equation for  $u$  is obtained by eliminating  $A$  and  $B$  from (22), (40), and (42). The total tide is

$$u + Ex^2.$$

## CASE 4.

21. *Sea bounded by two parallels of latitude on the same side of equator.* Suppose the sea to be bounded on the north and south by parallels of latitude and to lie entirely within one hemisphere.

It is necessary that, at the northern and southern boundaries,

$$\frac{du}{dx} + \frac{2}{x}u = 0.$$

Let the boundaries have colatitudes  $\theta_1, \theta_2$  ( $\theta_1 < \theta_2$ ), and let  $\sin \theta_1 = a_1, \sin \theta_2 = a_2$ .

Then the equations for the determination of  $A$  and  $B$  are similar to (42) and are

$$\left. \begin{aligned} A \left[ \varphi'(a_1) + \frac{2}{a_1} \varphi(a_1) + u_1'(a_1) \log a_1 + \frac{1}{a_1} u_1(a_1) + \frac{2}{a_1} u_1(a_1) \log a_1 \right] \\ + B \left[ u_1'(a_1) + \frac{2}{a_1} u_1(a_1) \right] + \vartheta'(a_1) + \frac{2}{a_1} \vartheta(a_1) = 0, \\ A \left[ \varphi'(a_2) + \frac{2}{a_2} \varphi(a_2) + u_1'(a_2) \log a_2 + \frac{1}{a_2} u_1(a_2) + \frac{2}{a_2} u_1(a_2) \log a_2 \right] \\ + B \left[ u_1'(a_2) + \frac{2}{a_2} u_1(a_2) \right] + \vartheta'(a_2) + \frac{2}{a_2} \vartheta(a_2) = 0. \end{aligned} \right\} \quad (43)$$

As before,

$$A [\varphi(x) + u_1(x) \log x] + Bu_1(x) + \vartheta(x) - u = 0. \quad (22)$$

Then, eliminating  $A$  and  $B$ , the equation for  $u$  is

$$\begin{aligned} & \varphi(x) + u_1(x) \log x, \quad u_1(x), \quad \vartheta(x) \\ & \varphi'(a_1) + \frac{2}{a_1} \varphi(a_1) + u_1'(a_1) \log a_1 + \frac{1}{a_1} u_1(a_1) + \frac{2}{a_1} u_1(a_1) \log a_1, \quad u_1'(a_1) + \frac{2}{a_1} u_1(a_1), \quad \vartheta'(a_1) + \frac{2}{a_1} \vartheta(a_1) \\ & \varphi'(a_2) + \frac{2}{a_2} \varphi(a_2) + u_1'(a_2) \log a_2 + \frac{1}{a_2} u_1(a_2) + \frac{2}{a_2} u_1(a_2) \log a_2, \quad u_1'(a_2) + \frac{2}{a_2} u_1(a_2), \quad \vartheta'(a_2) + \frac{2}{a_2} \vartheta(a_2) \\ & = \left[ \begin{aligned} & \varphi'(a_1) + \frac{2}{a_1} \varphi(a_1) + u_1'(a_1) \log a_1 + \frac{1}{a_1} u_1(a_1) + \frac{2}{a_1} u_1(a_1) \log a_1, \quad u_1'(a_1) + \frac{2}{a_1} u_1(a_1) \\ & \varphi'(a_2) + \frac{2}{a_2} \varphi(a_2) + u_1'(a_2) \log a_2 + \frac{1}{a_2} u_1(a_2) + \frac{2}{a_2} u_1(a_2) \log a_2, \quad u_1'(a_2) + \frac{2}{a_2} u_1(a_2) \end{aligned} \right] u. \quad (44) \end{aligned}$$

(44) gives the value of  $u$ . The complete tidal expression is

$$u + Ex^2.$$

The value of  $u$  at the boundary whose colatitude is  $\theta_1$  is given by

$$\begin{aligned} & \varphi'(a_1) + \frac{2}{a_1} \varphi(a_1) + u'(a_1) \log a_1 + \frac{1}{a_1} u_1(a_1) + \frac{2}{a_1} u_1(a_1) \log a_1, \quad u_1'(a_1) + \frac{2}{a_1} u_1(a_1) \\ & \varphi'(a_2) + \frac{2}{a_2} \varphi(a_2) + u'(a_2) \log a_2 + \frac{1}{a_2} u_1(a_2) + \frac{2}{a_2} u_1(a_2) \log a_2, \quad u_1'(a_2) + \frac{2}{a_2} u_1(a_2) \\ & \varphi(a_1), \quad u_1(a_1), \quad \vartheta(a_1) \\ = & \varphi'(a_1) + \frac{1}{a_1} u_1(a_1), \quad u_1'(a_1), \quad \vartheta'(a_1) \quad (44A) \\ & \varphi'(a_2) + \frac{2}{a_2} \varphi(a_2) + u_1'(a_2) \log \frac{a_2}{a_1} + \frac{1}{a_2} u_1(a_2) + \frac{2}{a_2} u_1(a_2) \log \frac{a_2}{a_1}, \quad u_1'(a_2) + \frac{2}{a_2} u_1(a_2), \quad \vartheta'(a_2) + \frac{2}{a_2} \vartheta(a_2) \end{aligned}$$

A similar equation may be obtained for the evaluation of  $u(a_2)$ .

#### CASE 5.

22. *Canal of width  $2d$  lying along a parallel of latitude.* Suppose the zonal sea to narrow down to a canal of width  $2d$  having as its northern boundary the parallel of latitude  $\frac{1}{2}\pi - \theta$ .

All three of the functions in the expression (22) are expressible in the neighborhood of  $\theta = \theta_1$  in series form. Let them be expressed in this manner and let  $d$  be taken so small that powers of it higher than the first may for purposes of calculation be neglected.

The second of the equations (43) then becomes, after simplification by means of the first one,

$$\begin{aligned} & A \left[ \varphi''(a_1) + \frac{2}{a_1} \varphi'(a_1) - \frac{2}{a_1^2} \varphi(a_1) + \frac{2}{a_1} u_1'(a_1) + \frac{1}{a_1^2} u_1(a_1) \right. \\ & \quad \left. + [u_1''(a_1) + \frac{2}{a_1} u_1'(a_1) - \frac{2}{a_1^2} u_1(a_1)] \log a_1 \right] \\ & + B \left[ u_1''(a_1) + \frac{2}{a_1} u_1'(a_1) - \frac{2}{a_1^2} u_1(a_1) \right] + \vartheta''(a_1) + \frac{2}{a_1} \vartheta'(a_1) - \frac{2}{a_1^2} \vartheta(a_1) \\ & + d(lA + mB + n) \sqrt{1 - a_1^2} = 0, \quad (45) \end{aligned}$$

where  $l, m, n$  may be easily found in terms of  $a_1$ .

Then the equation for the determination of  $u$  becomes

$$\begin{aligned} & \varphi(x) + u_1(x) \log x \\ & \varphi'(a_1) + \frac{2}{a_1} \varphi(a_1) + u_1'(a_1) \log a_1 + \frac{1}{a_1} u_1(a_1) + \frac{2}{a_1} u_1'(a_1) \log a_1 + \frac{2}{a_1} u_1(a_1) \\ & \varphi''(a_1) + \frac{2}{a_1} \varphi'(a_1) - \frac{2}{a_1^2} \varphi(a_1) + \frac{2}{a_1} u_1'(a_1) + \frac{1}{a_1^2} u_1(a_1) \left\{ u_1''(a_1) + \frac{2}{a_1} u_1'(a_1) - \frac{2}{a_1^2} u_1(a_1) \right\} \\ & + \log a_1 [u_1''(a_1) + \frac{2}{a_1} u_1'(a_1) - \frac{2}{a_1^2} u_1(a_1)] + d l_1 \sqrt{1 - a_1^2} \left\{ u_1''(a_1) + \frac{2}{a_1} u_1'(a_1) - \frac{2}{a_1^2} u_1(a_1) \right\} \\ & + d m \sqrt{1 - a_1^2} \left\{ u_1''(a_1) + \frac{2}{a_1} u_1'(a_1) - \frac{2}{a_1^2} u_1(a_1) \right\} = 0. \quad (46) \end{aligned}$$

This reduces to the form

$$\begin{aligned} & \varphi(x) + u_1(x) \log x \\ & \varphi'(a_1) + \frac{2}{a_1} \varphi(a_1) + \frac{1}{a_1} u_1(a_1) \\ & \varphi''(a_1) + \frac{2}{a_1} \varphi'(a_1) - \frac{2}{a_1^2} \varphi(a_1) + \frac{2}{a_1} u_1'(a_1) + \frac{1}{a_1^2} u_1(a_1) \left\{ u_1''(a_1) + \frac{2}{a_1} u_1'(a_1) - \frac{2}{a_1^2} u_1(a_1) \right\} \\ & + \frac{1}{a_1^2} u_1(a_1) + d(l - m \log a_1) \sqrt{1 - a_1^2} \left\{ u_1''(a_1) + \frac{2}{a_1} u_1'(a_1) - \frac{2}{a_1^2} u_1(a_1) \right\} \\ & + d m \sqrt{1 - a_1^2} \left\{ u_1''(a_1) + \frac{2}{a_1} u_1'(a_1) - \frac{2}{a_1^2} u_1(a_1) \right\} = 0, \end{aligned}$$

and again to

$$\begin{aligned} & \varphi(x) + u_1(x) \log x \\ & \varphi'(a_1) + \frac{2}{a_1} \varphi(a_1) + \frac{1}{a_1} u_1(a_1) \\ & \varphi''(a_1) + \frac{2}{a_1} \varphi'(a_1) + \frac{2}{a_1} u_1'(a_1) + \frac{1}{a_1^2} u_1(a_1) \left\{ u_1''(a_1) + \frac{2}{a_1} u_1'(a_1) - \frac{2}{a_1^2} u_1(a_1) \right\} \\ & + \frac{2}{a_1^2} u_1(a_1) + d(l - m \log a_1) \sqrt{1 - a_1^2} \left\{ u_1''(a_1) + \frac{2}{a_1} u_1'(a_1) - \frac{2}{a_1^2} u_1(a_1) \right\} \\ & + d m \sqrt{1 - a_1^2} \left\{ u_1''(a_1) + \frac{2}{a_1} u_1'(a_1) - \frac{2}{a_1^2} u_1(a_1) \right\} = 0. \quad (47) \end{aligned}$$

23. *Tide at point distant  $\delta$  from boundary of canal.* For any point within the canal distant  $\delta$  from the northern boundary  $x = a_1 + \delta \sqrt{1 - a_1^2}$ . This substitution being made and the product  $d\delta$  being neglected the result is

$$\begin{aligned}
 & \left| \begin{array}{ccc} \varphi(a_1) & , & u_1(a_1) & , & \vartheta(a_1) \\ \varphi'(a_1) + \frac{2}{a_1} \varphi(a_1) + \frac{1}{a_1} u_1(a_1) & , & u_1'(a_1) + \frac{2}{a_1} u_1(a_1) & , & \vartheta'(a_1) + \frac{2}{a_1} \vartheta(a_1) \\ \varphi''(a_1) + \frac{3}{a_1} \varphi'(a_1) + \frac{2}{a_1} u_1'(a_1) + \frac{2}{a_1^2} u_1(a_1) & , & u_1''(a_1) + \frac{3}{a_1} u_1'(a_1) & , & \vartheta''(a_1) + \frac{3}{a_1} \vartheta'(a_1) \end{array} \right| \\
 & + d \sqrt{1 - a_1^2} \left| \begin{array}{ccc} \varphi(a_1) & , & u_1(a_1) & , & \vartheta(a_1) \\ \varphi'(a_1) + \frac{2}{a_1} \varphi(a_1) + \frac{1}{a_1} u_1(a_1) & , & u_1'(a_1) + \frac{2}{a_1} u_1(a_1) & , & \vartheta'(a_1) + \frac{2}{a_1} \vartheta(a_1) \\ l - m \log a_1 & , & m & , & n \end{array} \right| \\
 & + \delta \sqrt{1 - a_1^2} \left| \begin{array}{ccc} \varphi'(a_1) + \frac{1}{a_1} u_1(a_1) & , & u_1'(a_1) & , & \vartheta'(a_1) \\ \varphi'(a_1) + \frac{1}{a_1} u_1(a_1) + \frac{2}{a_1} \varphi(a_1) & , & u_1'(a_1) + \frac{2}{a_1} u_1(a_1) & , & \vartheta'(a_1) + \frac{2}{a_1} \vartheta(a_1) \\ \varphi''(a_1) + \frac{3}{a_1} \varphi'(a_1) + \frac{2}{a_1} u_1'(a_1) + \frac{2}{a_1^2} u_1(a_1) & , & u_1''(a_1) + \frac{3}{a_1} u_1'(a_1) & , & \vartheta''(a_1) + \frac{3}{a_1} \vartheta'(a_1) \end{array} \right| \\
 & = u \left| \begin{array}{ccc} \varphi(a_1) + \frac{2}{a_1} \varphi(a_1) + \frac{1}{a_1} u_1(a_1) & , & u_1'(a_1) + \frac{2}{a_1} u_1(a_1) \\ \varphi'(a_1) + \frac{3}{a_1} \varphi'(a_1) + \frac{2}{a_1} u_1'(a_1) + \frac{2}{a_1^2} u_1(a_1) & , & u_1''(a_1) + \frac{3}{a_1} u_1'(a_1) \end{array} \right| \\
 & + u \sqrt{1 - a_1^2} d \left| \begin{array}{ccc} \varphi'(a_1) + \frac{2}{a_1} \varphi(a_1) + \frac{1}{a_1} u_1(a_1) & , & u_1'(a_1) + \frac{2}{a_1} u_1(a_1) \\ l - m \log a_1 & , & m \end{array} \right|. \quad (48)
 \end{aligned}$$

It is easy to show that

$$l - m \log a_1 = \varphi'''(a_1) + \frac{2}{a_1} \varphi''(a_1) - \frac{4}{a_1^2} \varphi'(a_1) + \frac{4}{a_1^3} \varphi(a_1)$$

$$+ \frac{3}{a_1} u_1''(a_1) + \frac{1}{a_1^2} u_1'(a_1) - \frac{4}{a_1^3} u_1(a_1),$$

$$m = u_1'''(a_1) + \frac{2}{a_1} u_1''(a_1) - \frac{4}{a_1^2} u_1'(a_1) + \frac{4}{a_1^3} u_1(a_1),$$

$$n = \vartheta'''(a_1) + \frac{2}{a_1} \vartheta''(a_1) - \frac{4}{a_1^2} \vartheta'(a_1) + \frac{4}{a_1^3} \vartheta(a_1).$$



Equation (48) can then be put in the form

$$\begin{aligned}
 & \left. \begin{aligned} & \varphi'(a_1) + \frac{2}{a_1} \varphi(a_1) + \frac{1}{a_1} u_1(a_1) \\ & \varphi''(a_1) + \frac{3}{a_1} \varphi'(a_1) + \frac{2}{a_1} u_1'(a_1) + \frac{2}{a_1^2} u_1(a_1) \end{aligned} \right\} , \left. \begin{aligned} & u_1'(a_1) + \frac{2}{a_1} u_1(a_1) \\ & u_1''(a_1) + \frac{3}{a_1} u_1'(a_1) \end{aligned} \right\} \\
 & + du \sqrt{1-a_1^2} \left\{ \begin{aligned} & \varphi'(a_1) + \frac{2}{a_1} \varphi(a_1) + \frac{1}{a_1} u_1(a_1) \\ & \varphi'''(a_1) + \frac{2}{a_1} \varphi''(a_1) + \frac{12}{a_1^3} \varphi(a_1) \\ & + \frac{3}{a_1} u_1''(a_1) + \frac{1}{a_1^2} u_1'(a_1) \end{aligned} \right\} , \left\{ \begin{aligned} & u_1'(a_1) + \frac{2}{a_1} u_1(a_1) \\ & u_1'''(a_1) + \frac{2}{a_1} u_1''(a_1) \\ & + \frac{12}{a_1^3} u_1(a_1) \end{aligned} \right\} \\
 & = \left. \begin{aligned} & \varphi(a_1) \\ & \varphi'(a_1) + \frac{1}{a_1} u_1(a_1) \\ & \varphi''(a_1) + \frac{2}{a_1} u_1'(a_1) - \frac{1}{a_1^2} u_1(a_1) \end{aligned} \right\} , \left. \begin{aligned} & u_1(a_1), \varphi'(a_1) \\ & u_1'(a_1), \varphi'(a_1) \\ & u_1''(a_1), \varphi''(a_1) \end{aligned} \right\} \left[ 1 + \frac{2(d-\delta)\sqrt{1-a_1^2}}{a_1} \right] \\
 & + d \sqrt{1-a_1^2} \left. \begin{aligned} & \varphi(a_1) \\ & \varphi'(a_1) + \frac{1}{a_1} u_1(a_1) \\ & \varphi'''(a_1) + \frac{3}{a_1} u_1''(a_1) - \frac{3}{a_1^2} u_1'(a_1) + \frac{2}{a_1^3} u_1(a_1) \end{aligned} \right\} , \left. \begin{aligned} & u_1(a_1), \varphi(a_1) \\ & u_1'(a_1), \varphi'(a_1) \\ & u_1'''(a_1), \varphi'''(a_1) \end{aligned} \right\} . \quad (49)
 \end{aligned}$$

The value of  $u$  at the middle of the canal is obtained by putting  $\delta = d$ , which simplifies (49) somewhat.

24. *Canal of negligible width.* If now the canal be taken so narrow that the width may be neglected, (49) is still further simplified. The results should coincide with those obtained by considering the motion as in two dimensions. In this case the equation for  $u$  is

$$\begin{aligned}
 & \left. \begin{aligned} & \varphi'(a_1) + \frac{2}{a_1} \varphi(a_1) + \frac{1}{a_1} u_1(a_1) \\ & \varphi''(a_1) + \frac{3}{a_1} \varphi'(a_1) + \frac{2}{a_1} u_1'(a_1) + \frac{2}{a_1^2} u_1(a_1) \end{aligned} \right\} , \left. \begin{aligned} & u_1'(a_1) + \frac{2}{a_1} u_1(a_1) \\ & u_1''(a_1) + \frac{3}{a_1} u_1'(a_1) \end{aligned} \right\} \\
 & = \left. \begin{aligned} & \varphi(a_1) \\ & \varphi'(a_1) + \frac{1}{a_1} u_1(a_1) \\ & \varphi''(a_1) + \frac{2}{a_1} u_1'(a_1) - \frac{1}{a_1^2} u_1(a_1) \end{aligned} \right\} , \left. \begin{aligned} & u_1(a_1), \varphi'(a_1) \\ & u_1'(a_1), \varphi'(a_1) \\ & u_1''(a_1), \varphi''(a_1) \end{aligned} \right\} . \quad (50)
 \end{aligned}$$

As before, the total tide is

$$U = u + E a_1^2. \quad (51)$$

### VIII.

#### SUMMARY OF RESULTS.

25. *Summary of I-IV.* For convenience of reference, and in order to render the results available to any who do not desire to follow through the processes of obtaining them, it has been thought desirable that they should be restated in a separate section which, along with the historical sketch in Section II, would give a complete account of the state of the problem. Section III is devoted to a discussion and criticism of the analytical process by means of which Laplace obtained his value for the arbitrary constant in his solution. Objection is taken to the process employed for two reasons. In the first place, the reasoning used has not been shown to be, and does not appear to be strictly accurate. In this connection it may be said that in the examination of the apparent inaccuracies it has been thought sufficient to indicate the weaknesses of the method rather than go into a minute discussion of them. The modern advances in the theory of Differential Equations make it appear probable that matters of this character will be treated differently in future. In Section IV the complete solution of the equation is found, the expressions involved being infinite series, whose regions of convergence are large enough to make possible the treatment of all cases that can arise. The regions of convergence of the series together with certain important properties of the integrals can be predicted from the form of the equation. The integral found is more general than that previously deduced involving the two arbitrary constants. The series used by Laplace enters this integral as a part of it. In the derivation of the integral the method of Laplace as given in the *Mécanique Céleste* is made clear. In the closing paragraphs of the section certain properties of the integral important in the application of the physical conditions are deduced.

26. *Summary of V-VII.* Section V deals with the applications of the physical conditions to the determination of the arbitrary constants. One of the constants is immediately determined. The determination of the other involves greater difficulties. The analysis on which previous evaluations have been based is rejected and replaced by a determination of the value which appears to be entirely satisfactory. The condition which determines this constant is the condition stated by Lord Kelvin. The result of this section appears to be a complete justification of Laplace's series.

In Section VI the objections taken to the proof previously given in the determination of the second arbitrary constant are set forth.

The last Section contains the discussion of five important cases in the theory of tides. The arbitrary constants in the general integral of the differential equation are determined so that the integrals represent the tidal disturbance in these cases, and expressions are obtained for the tidal disturbance at any point whatever, and at certain particular points, such as points on the boundary. The last of the five cases treated is that of a canal lying along a parallel of latitude and would appear to furnish a means of checking the same case treated by Airy's Canal Theory of Tides.

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